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LAMINAR FORCED CONVECTION IN RECTANGULAR CHANNELS WITH UNEQUAL HEAT ADDITION ON ADJACENT SIDES

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ABSTRACT

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Temperature distributions are determined analytically for fully developed laminar heat transfer in channels with aspect ratios from 1 to ∞. The channel walls are uniformly heated, but the heat flux on the short sides is an arbitrary fraction between 0 and 1 of the heat flux on the broad sides. For all cases, the wall temperatures are compared on the basis that the total heat transferred per unit channel length is maintained at a fixed value. The poor convection due to the low velocities in the corners and along the narrow walls always caused the peak temperatures to occur at the corners. The lowest peak temperatures were found when all the heating took place at only the broad walls rather than when heating was partly distributed to the short sides. This results from the fact that, when four sides are heated, more energy is being supplied to the low velocity corner regions. For heating at only the broad walls, the corner temperature decreases rapidly as the aspect ratio is increased to about 10 and insignificantly thereafter. In the limit of infinite aspect ratio, the wall temperature distribution does not approach a constant as is the case for infinite parallel plates.

AUTHOR

NOMENCLATURE

a,b half lengths of short and broad sides, respectively

Q	heat addition per unit channel length
q	heat addition per unit wall area
T	temperature
u	fluid velocity
$\bar{u}$	mean fluid velocity
X	dimensionless coordinate, x/a
x	coordinate measured along short side from channel center
Y	dimensionless coordinate, y/b
y	coordinate measured along long side from channel center
β	heat flux ratio, q_S/q_B
γ	aspect ratio, b/a
θ	dimensionless temperature, $4kT/Q$

Subscripts:

B	refers to broad side
b	bulk mean value
S	refers to short side
w	value at wall

INTRODUCTION

Rectangular coolant channels are often employed in heat-exchange devices, particularly in nuclear reactor plate-type fuel assemblies where wide, parallel fuel-bearing plates are supported by unfueled side plates. In such assemblies, most of the total heating is produced in the broad, fueled plates with the remainder (usually less than 10 percent) resulting from gamma heating in the support walls. For example, in [1] 3 percent of the total heating is generated in the support walls. Cooling is accomplished by passing high velocity fluid through the channels. A factor of importance for proper operation of the reactor is maintaining a satisfactory temperature distribution in the cooling channel walls.

Several papers have treated laminar fully developed heat transfer in rectangular channels. In [2] the problem is examined where both uniform and nonuniform heating take place on a large fraction of only the broad walls. As part of the solution in [2], the case of uniform heating over the entire broad walls (with the side walls unheated) was solved numerically for channels having aspect ratios of 10 and 20. In [3], variational methods were described for channel heat-transfer problems, and results were evaluated for aspect ratios of 1 and 10 for uniform heat flux on all four sides, and for an aspect ratio of 10 with uniform heating on the broad sides only. An analytical solution was obtained in [4] for uniform heating on four sides, and results were evaluated for aspect ratios of 1, 2, and 4.

It is the purpose of this note to provide the general solution where heating occurs on all four walls for the conditions that the uniform heat flux on the short walls is any prescribed fraction between 0 and 1 of the flux on the broad walls. The total heat input per unit channel length is maintained constant, and aspect ratios from 1 to ∞ are considered.

ANALYSIS

The rectangular channel and its coordinate system are shown in Fig. 1. Only the fully developed velocity and temperature regions are considered, and the fluid is assumed to have constant properties.

<u>Velocity distribution</u>. - For steady laminar flow, the velocity

distribution u, as given in [5], was integrated over the cross section and the result divided by the cross-sectional area to give the mean velocity $\bar{u}$. This was used to nondimensionalize u:

$$\frac{u}{\bar{u}} = G\left\{-\frac{\gamma}{8} + \frac{\gamma X^2}{8} - \frac{4\gamma}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{(n+1)}{2}}}{n^3 \cosh\left(\frac{n\pi\gamma}{2}\right)}\left[\cosh\left(\frac{n\pi\gamma}{2}Y\right)\cos\left(\frac{n\pi X}{2}\right)\right]\right\} \tag{1}$$

where

$$G = \left[-\frac{\gamma}{12} + \frac{16}{\pi^5}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^5}\tanh\left(\frac{m\pi\gamma}{2}\right)\right]^{-1}$$

γ is the aspect ratio b/a, $X = x/a$, and $Y = y/b$.

Energy equation. - The energy equation for the fluid temperature T, with viscous dissipation neglected, is

$$\rho c_p u \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{2}$$

where ρ, c_p, and k are the fluid density, specific heat, and thermal conductivity, respectively. Under the assumptions of constant heat addition per unit channel length, Q, and a fully developed temperature profile, a heat balance on a unit of channel length provides the result:

$$\frac{\partial T}{\partial z} = \frac{Q}{4ab\rho c_p \bar{u}}, \qquad \frac{\partial^2 T}{\partial z^2} = 0$$

Equation (2) now becomes

$$\frac{u}{\bar{u}}\left(\frac{Q}{4ab}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{3}$$

From symmetry, only the first quadrant need be considered, and the boundary conditions are

$$\left.\begin{aligned} 0 \le x \le a: &\quad \frac{\partial T}{\partial y}(x,0) = 0; \quad \frac{\partial T}{\partial y}(x,b) = \frac{Q}{4k\left[\frac{b}{\beta} + a\right]} = \frac{q_S}{k} \\ 0 \le y \le b: &\quad \frac{\partial T}{\partial x}(0,y) = 0; \quad \frac{\partial T}{\partial x}(a,y) = \frac{Q}{4k[b + a\beta]} = \frac{q_B}{k} \end{aligned}\right\} \qquad (4)$$

where $\beta = q_S/q_B$, a parameter in the problem, in which q_B and q_S are, respectively, the heat fluxes at the broad and short walls. The energy Eq. (3) is to be solved subject to the boundary conditions (4) using the velocity distribution (1).

Superposition of solutions. - Equation (3) is written in terms of the dimensionless temperature $\theta = 4kT/Q$:

$$\nabla^2\theta \equiv \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{1}{ab}\,\frac{u}{\bar{u}} \qquad (3a)$$

This is a nonhomogeneous second order partial differential equation, and the difficulty of solution is caused by the complexity of the $u/\bar{u}$ term. By superposition, the solution is expressed as the sum of a particular solution θ_p, which satisfies the Poisson equation:

$$\nabla^2\theta_p = \frac{1}{ab}\,\frac{u}{\bar{u}} \qquad (5a)$$

and a complementary solution θ_c, which satisfies the Laplace equation:

$$\nabla^2\theta_c = 0 \qquad (5b)$$

A particular solution can be adapted from the one given in [4] and is written in the dimensionless form:

$$\theta_p = \theta_\pi + \theta^* = G\left\{\frac{X^4}{96} - \frac{\gamma^2 Y^2}{16} - \frac{8}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{(n+1)}{2}}}{n^5 \cosh\left(\frac{n\pi\gamma}{2}\right)}\right.$$

$$\left.\times\left[\cosh\left(\frac{n\pi\gamma}{2}Y\right) + \left(\frac{n\pi\gamma}{2}\right)Y\sinh\left(\frac{n\pi\gamma}{2}Y\right)\right]\cos\left(\frac{n\pi X}{2}\right)\right\} + C^*(X^2 - \gamma^2Y^2) \qquad (6)$$

where G was given after Eq. (1) and $C^* = 1/2[\gamma + \beta]$.

The complementary solution is divided into two parts $\theta_c = \theta_1 + \theta_2$ having the boundary condition

$$\left.\begin{aligned} &\frac{\partial\theta_1}{\partial x}(0,y) = \frac{\partial\theta_1}{\partial y}(x,0) = \frac{\partial\theta_1}{\partial y}(x,b) = 0 \\ &\frac{\partial\theta_1}{\partial x}(a,y) = \frac{1}{b+2\beta} - \frac{2C^*}{a} - \frac{\partial\theta_\pi}{\partial x}(a,y) \end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned} &\frac{\partial\theta_2}{\partial x}(0,y) = \frac{\partial\theta_2}{\partial x}(a,y) = \frac{\partial\theta_2}{\partial y}(x,0) = 0 \\ &\frac{\partial\theta_2}{\partial y}(x,b) = \frac{1}{\frac{b}{\beta}+a} + \frac{2C^*b}{a^2} - \frac{\partial\theta_\pi}{\partial y}(x,b) \end{aligned}\right\} \qquad (8)$$

In addition to satisfying these boundary conditions, it is a restriction of the Neumann problem for Laplace's equation that the line integral of the normal derivatives around the boundary be zero, a condition that was used to evaluate C^* from either Eqs. (7) or Eqs. (8). It is the necessity of satisfying these line integral conditions that required the θ^* function to be introduced.

The solutions for θ_1 and θ_2 were found by using product solutions in conjunction with Fourier series expansions of the boundary conditions.

The final dimensionless forms are:

$$\theta_1 = \sum_{n=1,2,3,\ldots}^{\infty} A_n \frac{\cosh\left(\frac{n\pi}{\gamma} X\right)}{\sinh\left(\frac{n\pi}{\gamma}\right)} \cos(n\pi Y)$$

$$A_n = G\left\{\frac{4\gamma^2(-1)^n}{\pi^6 n}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^3\left[\left(\frac{m\gamma}{2}\right)^2 + n^2\right]}\left[\frac{m\pi\gamma}{2} + \frac{2n^2}{\left(\frac{m\gamma}{2}\right)^2 + n^2}\tanh\left(\frac{m\pi\gamma}{2}\right)\right]\right\} \tag{9}$$

$$\theta_2 = \sum_{n=1,2,3,\ldots}^{\infty} B_n \frac{\cosh(n\pi\gamma Y)}{\sinh(n\pi\gamma)} \cos(n\pi X)$$

$$B_n = G\left\{\frac{4(-1)^n}{\pi^6 n}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^3\left[n^2 - \left(\frac{m}{2}\right)^2\right]}\left[2\tanh\left(\frac{m\pi\gamma}{2}\right) + \frac{m\pi\gamma}{2}\right]\right\} \tag{10}$$

The analytical forms of θ_1 and θ_2 automatically satisfy the zero derivative conditions in Eqs. (7) and (8). The Fourier coefficients A_n and B_n were evaluated to satisfy the finite derivative conditions $\partial\theta_1(a,y)/\partial x$ and $\partial\theta_2(x,b)/\partial y$, respectively. It is significant to note that the constant terms, $\left(\frac{1}{b + a\beta} - \frac{2C^*}{a}\right)$ and $\left(\frac{1}{\frac{b}{\beta} + a} + \frac{2C^*b}{a^2}\right)$, in these boundary conditions do not make a contribution to the A_n and B_n, as they are multiplied by $\cos(n\pi Y)$ and $\cos(n\pi X)$, respectively, and are integrated from 0 to 1 resulting in zero values. Hence, the θ_1 and θ_2

solutions depend only on the θ_π part of the particular solution, and along with θ_π, are independent of β. As a result, it is the β factor in the θ^* function that alone accounts for the unequal heat fluxes on adjacent sides.

Bulk temperature. - The solution $\theta = \theta_\pi + \theta^* + \theta_1 + \theta_2$ is substituted into the definition for the bulk temperature:

$$\theta_b = \frac{1}{ab} \int_0^a \int_0^b \frac{u}{\bar{u}} \theta \, dx \, dy \tag{11}$$

The integrations are carried out, and after considerable algebraic manipulation the bulk temperature is given by

$$\theta_b = \frac{\gamma}{\gamma + \beta} \left(\frac{G}{40} + \frac{G\gamma^2}{72} \right) + \frac{G^2\gamma^3}{576} + \frac{131G^2\gamma}{40320}$$

$$+ \frac{G}{4\pi^3} \sum_{n=1,2,3,\ldots}^{\infty} B_n \frac{(-1)^n}{n^3} + \frac{4G^2\gamma}{\pi^8} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^8 \cosh^2\left(\frac{m\pi\gamma}{2}\right)}$$

$$+ G \sum_{n=1,3,5,\ldots}^{\infty} \tanh\left(\frac{n\pi\gamma}{2}\right) \left\{ \frac{1}{(n\pi)^5} \left[\frac{\gamma}{\gamma + \beta} \left(\frac{8}{\gamma} - 8\gamma \right) - G\gamma^2 + \frac{G}{6} - \frac{128}{(\gamma + \beta)(n\pi)^2} \right.\right.$$

$$\left.\left. - \frac{16G}{(n\pi)^2} + \frac{72G}{(n\pi)^4} \right] \right\} + \frac{G}{\pi^5} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} \frac{(-1)^n A_n}{m\left[\left(\frac{m}{2}\right)^2 + \left(\frac{n}{\gamma}\right)^2\right]^2} \frac{\tanh\left(\frac{m\pi\gamma}{2}\right)}{\tanh\left(\frac{n\pi}{\gamma}\right)}$$

$$\times \frac{2G}{\pi^5} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{j=1,2,3,\ldots}^{\infty} \frac{(-1)^j B_j}{m^2\left[\left(\frac{m}{2}\right)^2 - j^2\right]^2} \left[\frac{m \tanh\left(\frac{m\pi\gamma}{2}\right)}{2 \tanh j\pi\gamma} - j \right] \tag{12}$$

The final solution is then given as $\theta - \theta_b = \theta_\pi + \theta^* + \theta_1 + \theta_2 - \theta_b$ in which γ and β are the only parameters.

LIMITING CASE OF INFINITE ASPECT RATIO

The solution was examined for the case where $\gamma \to \infty$ resulting in the following limiting analytical expression for the wall temperature distribution:

$$\theta(1,Y) - \theta_b = \left(\frac{96Y^2 - 32}{\pi^5}\right)\left(\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5}\right) + \frac{\beta}{2} Y^2 - \frac{\beta}{6} \tag{13}$$

$$\theta(X,1) - \theta_b = \frac{\beta}{3} + \frac{64}{\pi^5} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tag{14}$$

where (13) and (14) agree at the corner $X = 1$, $Y = 1$, and

$$\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} = \frac{31}{32}\,(1.03693)$$

in which the last number is the Riemann zeta function of argument 5.

DISCUSSION

Wall temperature distributions were evaluated from the analytical solutions and are presented in Figs. 2 to 4. At some places along the wall, the value of $\dfrac{T_w - T_b}{Q/4k}$ is negative, which means that T_b is larger than T_w. This may seem to contradict the fact that heat is flowing from the wall to the fluid. However, it must be recalled that T_b is an average value over the entire cross section, while T_w is a local value along the wall.

Because of the complexity of the equations, the influence of the aspect ratio γ and the heating ratio parameter β are not readily explained directly from the analytical form of the solution. For this reason, an attempt is made to present physically plausible explanations for some of the trends in the graphical results. The local temperature differences $T_w - T_b$ depend only on the quantities k, Q, β, and γ, and the physical significance of the β parameter should be kept in mind when interpreting the figures. When $\beta = 0$, there is no heat being transferred from the short sides, so all the Q is uniformly transferred from the broad walls only. For a fixed aspect ratio, as β is raised, an increasing portion of the Q is transferred from the short sides with a corresponding decrease in the total heat leaving the broad sides. When $\beta = 1$, all portions of the periphery have the same local heat flux.

Fig. 2 provides results, for various aspect ratios, for the important cases where either only two walls or all four walls are uniformly heated. Consider first the set of solid lines for $\beta = 0$, where all of the Q is being dissipated from only the broad walls. It is convenient to consider a set of channels having the same width 2b, the same total Q, and with the aspect ratio being increased by diminishing the spacing 2a. Under these conditions, the heat flux on the broad sides will remain constant. In a square duct, $\gamma = 1$, the heat flow paths from the two heated walls toward the region of high velocity fluid are approximately equal for all positions on the heated side, and hence, the temperature along these walls is almost uniform. As γ is increased (by decreasing the spacing, 2a) the heated wall temperatures decrease because, for narrower ducts, the paths for heat flowing to the region of higher velocities are shortened.

These paths are roughly perpendicular to the broad walls everywhere except for heat flowing from the corner regions. Because of the low velocities near the side walls when γ is large, some of the corner energy must be conducted within the fluid through a longer path in a direction parallel to the broad sides to reach a higher velocity region where it can be carried away more easily. Although the region of low velocity fluid occupies proportionately less of the cross-sectional area as γ increases, the width of the heat conduction path in the direction parallel to the heated side is also decreased as the duct becomes more narrow. As a result, when $\gamma \to \infty$, the temperature distribution along the heated wall goes to a limit with a maximum in the corner (Eqs. (13) and (14)) rather than to a uniform temperature, as is the case for a duct of infinite parallel plates without bounding side walls. The reason is that there must always be a spanwise temperature gradient to transport the imposed heating away from the low velocity region of the corners and side walls.

Now consider the set of dotted curves for $\beta = 1$ in Fig. 2. In this instance there is uniform heating all around the duct periphery, and the corner regions receive heat from two sides rather than from one, as was the case for $\beta = 0$. The interesting consequence is that the peak temperature remains essentially constant as γ is increased. However, the temperature gradients along the broad sides again increase with γ to remove the heat from the corner and side wall regions.

In Fig. 3 is shown the effect of changing β between 0 and 1 for two extremes in aspect ratio, $\gamma = 1$ and 20. As β is increased, the shifting of heat to the wall, which had been unheated for $\beta = 0$, tends to increase its temperature, while the temperature of the wall, which had received all the heat when $\beta = 0$, tends to decrease. For a square duct, $\gamma = 1$ (dotted lines), the corner temperatures remain fixed. In this case, although changing β shifts heat from one side to the other, the heating received by the corner region remains constant, because the heat that is removed from one wall that forms the corner is added through the other. Now consider the results for $\gamma = 20$ in Fig. 3 (solid lines). As β goes from 0 to 1, it is necessary to remove only a small amount of the total Q from the broad walls and shift it to the short walls to provide the same uniform heat flux at the short wall. The low velocity region then receives a little less heat from the broad walls, but this decrease is much less than the additional heat it receives from the short walls. The result is an increase of both the spanwise temperature gradient and the corner temperature as β is increased, a result which was found to hold true for all rectangular ducts ($\gamma > 1$). Thus, these results lead to the <u>important conclusion</u> that it is better to transfer all the heat through the broad walls of rectangular channels than to distribute it around the entire periphery (under the restrictions that the walls are nonconducting and that the heating extends all the way into the corners).

The corner temperatures are of practical importance because that is where the maximum wall temperatures occur for all aspect ratios and values of β. In Fig. 4 are shown the peak temperatures as a function of the aspect ratio for various β values. The largest reduction in the corner temperatures of rectangular channels is achieved when all the heat is transferred through the broad walls only ($\beta = 0$) and when the aspect ratio is increased to about 10. Beyond $\gamma \approx 10$, only a small reduction occurs, so that for rectangular cooling channels in nuclear reactors, where $\beta \approx 0.03$, the optimum aspect ratio appears to be about 10 to 20. If the side wall heating is increased ($\beta > 0$), the corner temperatures do not drop off as rapidly with larger γ as for the case where $\beta = 0$. When the heat flux is very nearly uniform over all walls ($0.75 < \beta < 1$), the peak temperatures are almost constant for all aspect ratios. Hence, it is concluded that for many nuclear reactors that utilize rectangular channels, it would be a disadvantage to load the narrow side walls with fuel even if it were possible. The advantage gained by the increased heat-transfer area would be offset by the additional heating imposed near the corner resulting in higher corner temperatures and higher maximum to minimum wall temperature differences.

ACKNOWLEDGMENT

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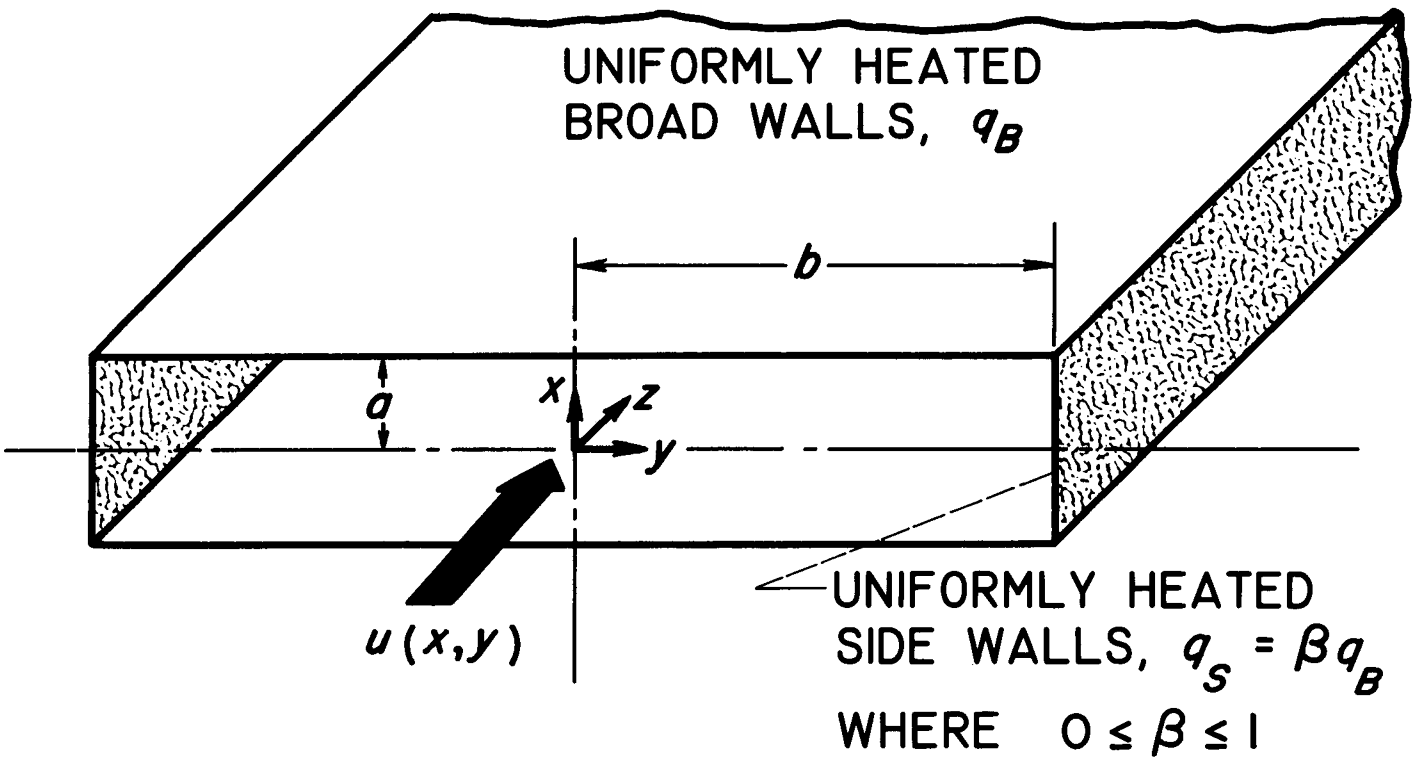

Fig. 1. - Coordinate system for rectangular channel with different uniform heating on each pair of opposite walls, $Q = 4aq_S + 4bq_B$.

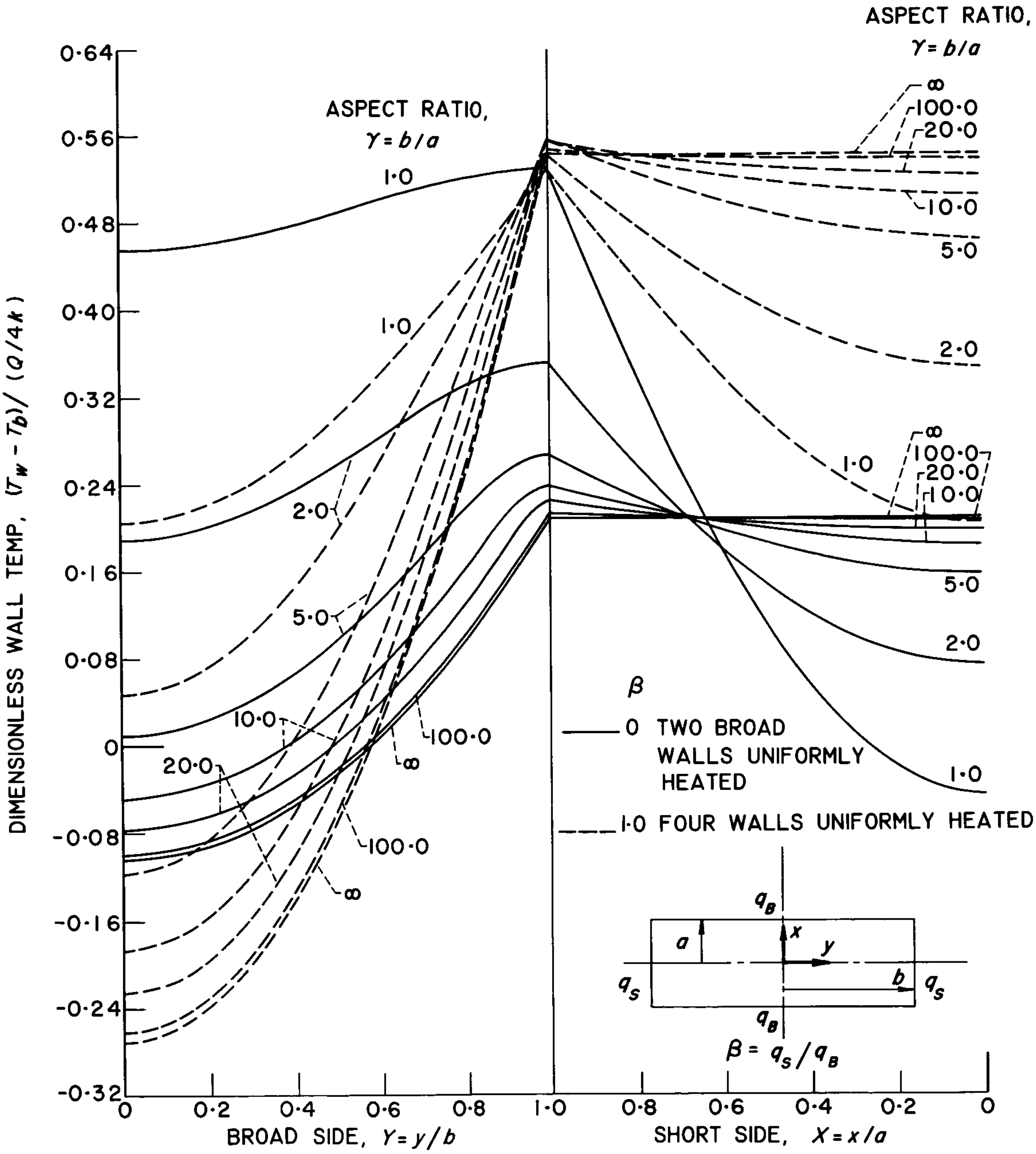

Fig. 2. - Wall temperatures of rectangular channels for cases of uniform heat flux on two walls and uniform heat flux on four walls.

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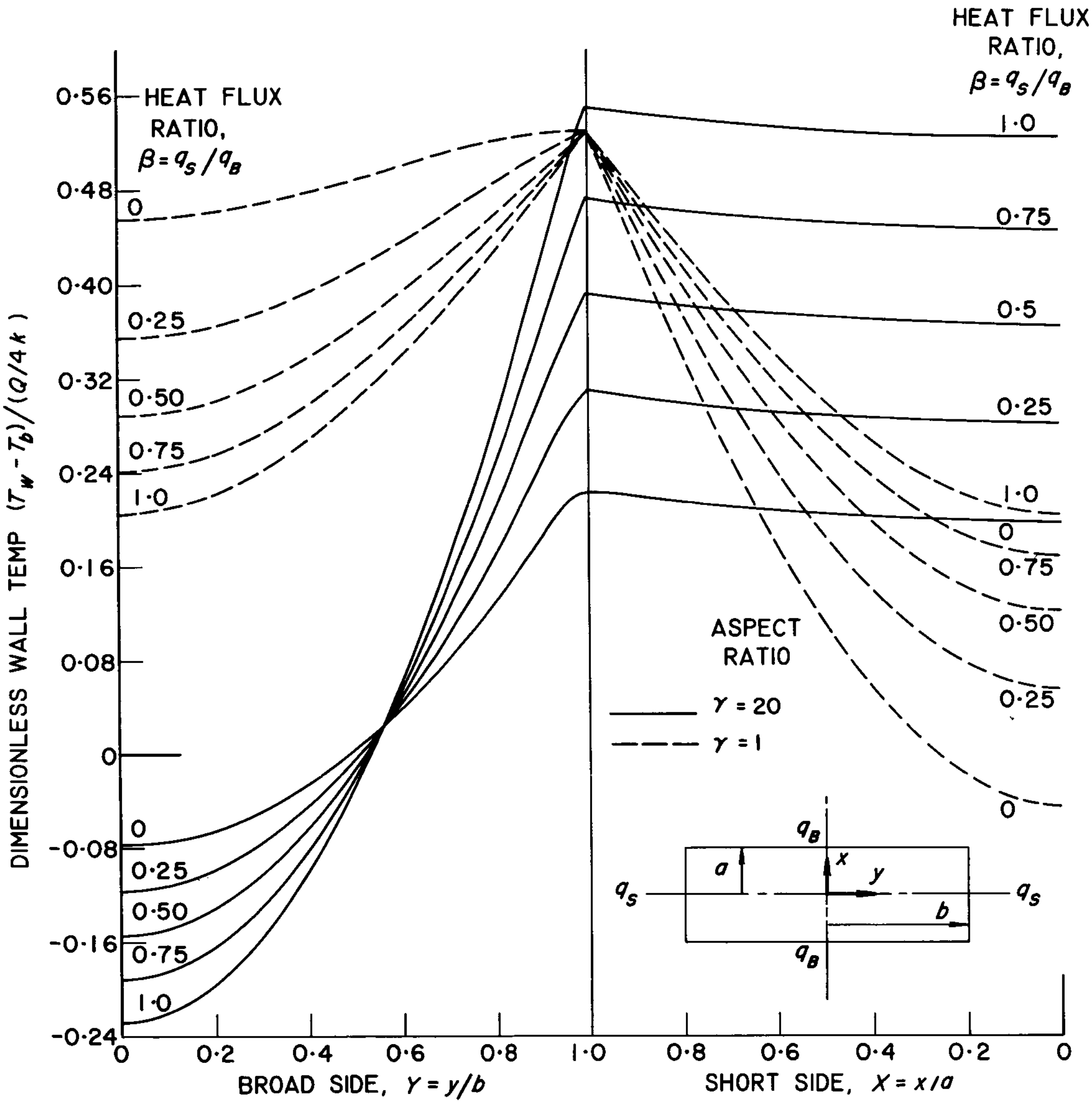

Fig. 3. - Wall temperatures for various ratios, β, of the heat flux on the short walls to that on the broad walls in channels with aspect ratios γ = 1 and 20.

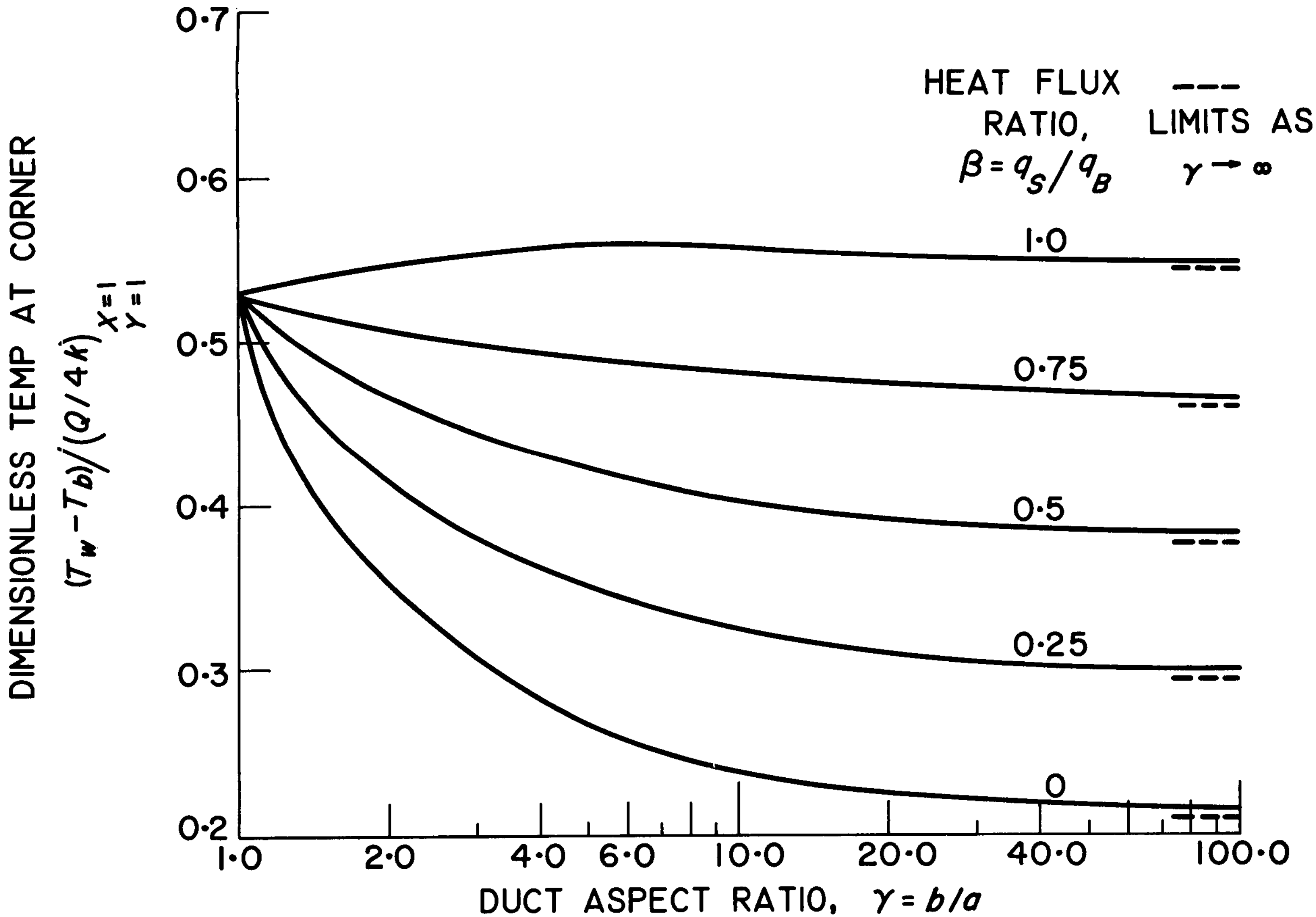

Fig. 4. - Dependence of temperature at duct corner on aspect ratio, γ, for various values of heating distribution parameter, β.